Applied Machine Learning

Gradient Computation & Automatic Differentiation

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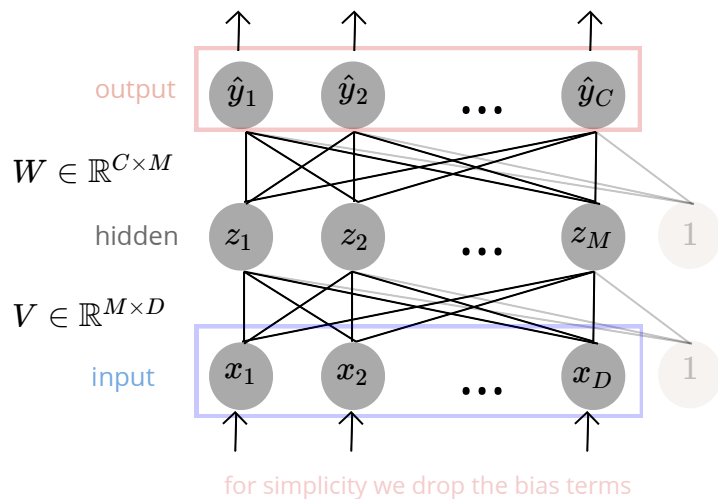
Learning objectives

using the chain rule to calculate the gradients

automatic differentiation

- forward mode
- reverse mode (backpropagation)
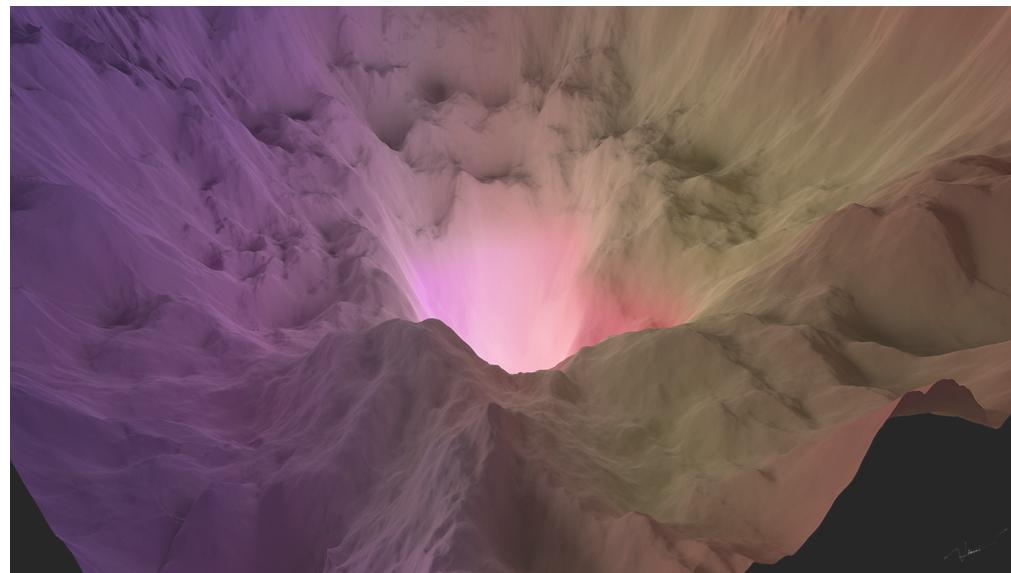
Landscape of the cost function

model   two layer MLP

$$f(x; W, V) = g(Wh(Vx))$$

output $\hat{y}_1$ $\hat{y}_2$ ... $\hat{y}_C$

$W \in \mathbb{R}^{C \times M}$

hidden z_1 z_2 ... z_M 1

$V \in \mathbb{R}^{M \times D}$

input x_1 x_2 ... x_D 1

for simplicity we drop the bias terms

objective $\min_{W,V} \sum_n L(y^{(n)}, f(x^{(n)}; W, V))$

loss function depends on the task

this is a non-convex optimization problem

https://losslandscape.com/gallery/
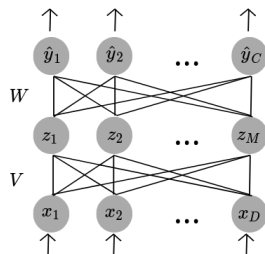
Landscape of the cost function

model two layer MLP

$$f(x; W, V) = g\big(Wh(Vx)\big)$$

there are **exponentially many** optima
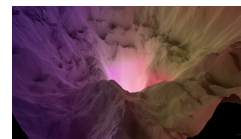
given one optimum V*, W* we can create many more with the same cost:

- permute hidden units in each layer (M!) weight space symmetry
- for symmetric activations: negate input/ouput of a unit
- for ReLU: rescale input/output weights attached to a unit
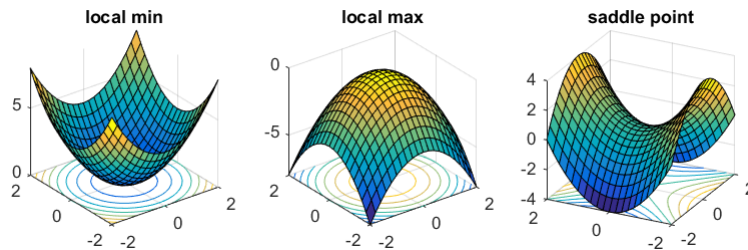
objective $\min_{W,V} \sum_n L(y^{(n)}, f(x^{(n)}; W, V))$
loss function depends on the task

this is a non-convex optimization problem

many critical points (points where gradient is zero)

local min   local max   saddle point

these are not stable and SGD can escape

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Landscape of the cost function

there are **exponentially many** optima

given one optimum V*, W* we can create many more with the same cost:

- permute hidden units in each layer (M!)
- for symmetric activations: negate input/ouput of a unit
- for ReLU: rescale input/output weights attached to a unit

this is a non-convex optimization problem

general beliefs

supported by empirical and theoretical results in a special settings

many more saddle points than local minima
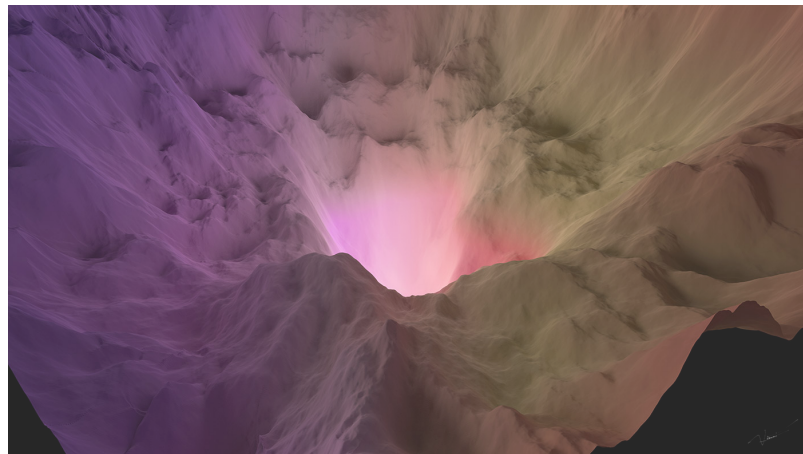
number of local minima increases for lower costs

therefore most local optima are close to global optima

strategy   use gradient descent methods

(covered earlier in the course)

https://losslandscape.com/gallery/

Jacobian matrix

$f : \mathbb{R} \to \mathbb{R}$ we have the derivative $\frac{d}{dw} f(w) \in \mathbb{R}$

$f : \mathbb{R}^D \to \mathbb{R}$ **gradient** is the vector of all partial derivatives

$$\nabla_w f(w) = [\tfrac{\partial}{\partial w_1} f(w), \dots, \tfrac{\partial}{\partial w_D} f(w)]^\top \in \mathbb{R}^D$$

$f(x,y) = x^2 + y^2 \Rightarrow$
$\nabla_{w=[x,y]} f(w) = [2x, 2y]$

$f : \mathbb{R}^D \to \mathbb{R}^M$ the **Jacobian matrix** of all partial derivatives

$$\frac{\partial}{\partial w_1} f(w)$$

$f(x,y) = [x^2, y^2, 2xy] \Rightarrow$

$$J = \begin{bmatrix} \dfrac{\partial f_1(w)}{\partial w_1}, & \cdots, & \dfrac{\partial f_1(w)}{\partial w_D} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_M(w)}{\partial w_1}, & \cdots, & \dfrac{\partial f_M(w)}{\partial w_D} \end{bmatrix} \in \mathbb{R}^{M \times D}$$

$\nabla_w f_1(w)$

$$J = \begin{bmatrix} 2x, & 0 \\ 0, & 2y \\ 2y, & 2x \end{bmatrix}$$

note that we use J also for cost function

$$J_{ij} = \frac{\partial f_i(w)}{\partial w_j}$$

for all three case we may simply write $\frac{\partial}{\partial w} f(w)$, where M,D will be clear from the context

what if W is a matrix? we assume it is reshaped into a vector for these calculations

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Chain rule

for $f : x \mapsto z$ and $h : z \mapsto y$ where $x, y, z \in \mathbb{R}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

speed of change in z as we change x

speed of change in y as we change z

speed of change in y as we change x

more generally $x \in \mathbb{R}^D, z \in \mathbb{R}^M, y \in \mathbb{R}^C$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial z}\frac{\partial z}{\partial x} \quad \text{in matrix form}$$

C x D Jacobian

M x D Jacobian

C x M Jacobian

$$\frac{\partial y_c}{\partial x_d} = \sum_{m=1}^{M} \frac{\partial y_c}{\partial z_m}\frac{\partial z_m}{\partial x_d}$$

we are looking at all the "paths" through which change in x_d changes y_c and add their contribution

$\hat{y}_1$ $\hat{y}_2$ ... $\hat{y}_C$

z_1 z_2 ... z_M

x_1 x_2 ... x_D
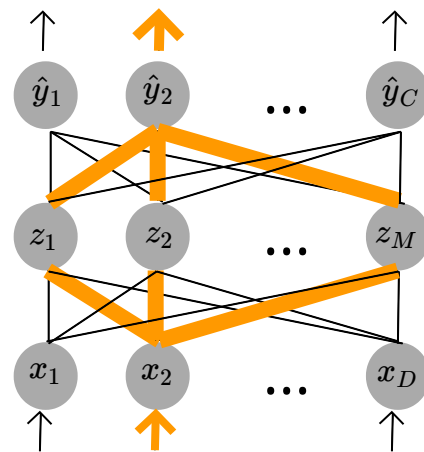
Training a two layer network

model $\quad \hat{y} = g\big(W\,h(V\,x)\big)$

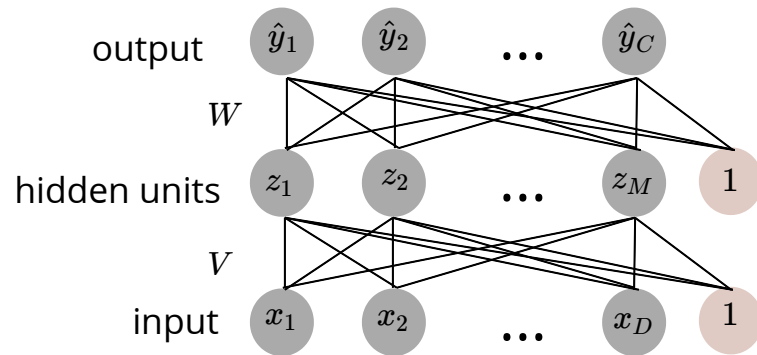Cost function   we want to minimize

$$J(W, V) = \sum_n L(y^{(n)}, g\,(\,W\,h\,(\,V\,x^{(n)}\,)\,))$$

need gradient wrt W and V: $\frac{\partial}{\partial W}J,\ \frac{\partial}{\partial V}J$

simpler to write this for one instance (n)

so we will calculate $\ \frac{\partial}{\partial W}L,\ \frac{\partial}{\partial V}L\ $ and recover

$$\frac{\partial}{\partial W}J = \sum_{n=1}^{N}\frac{\partial}{\partial W}L(y^{(n)}, \hat{y}^{(n)}) \ \text{ and }\ \frac{\partial}{\partial V}J = \sum_{n=1}^{N}\frac{\partial}{\partial V}L(y^{(n)}, \hat{y}^{(n)})$$

output $\quad \hat{y}_1 \quad \hat{y}_2 \quad \cdots \quad \hat{y}_C$

W

hidden units $\quad z_1 \quad z_2 \quad \cdots \quad z_M \quad 1$

V

input $\quad x_1 \quad x_2 \quad \cdots \quad x_D \quad 1$

for simplicity we drop the bias terms

Gradient calculation

using the chain rule

☺ $$\frac{\partial}{\partial W_{c,m}} L = \frac{\partial L}{\partial \hat{y}_c} \frac{\partial \hat{y}_c}{\partial u_c} \frac{\partial u_c}{\partial W_{c,m}}$$

depends on the loss function

z_m

depends on the activation function

similarly for V

☺ $$\frac{\partial}{\partial V_{m,d}} L = \sum_c \frac{\partial L}{\partial \hat{y}_c} \frac{\partial \hat{y}_c}{\partial u_c} \frac{\partial u_c}{\partial z_m} \frac{\partial z_m}{\partial q_m} \frac{\partial q_m}{\partial V_{m,d}}$$

depends on the loss function

$W_{c,m}$ x_d

depends on the activation function   depends on the middle layer activation

$L(y, \hat{y})$

$\hat{y}_c = g(u_c)$

pre-activations $u_c = \sum_{m=1}^{M} W_{c,m} z_m$

$z_m = h(q_m)$

pre-activations $q_m = \sum_{d=1}^{D} V_{m,d} x_d$

x_d

Gradient calculation

using the chain rule

☺ $$\frac{\partial}{\partial W_{c,m}} L = \frac{\partial L}{\partial \hat{y}_c} \frac{\partial \hat{y}_c}{\partial u_c} \frac{\partial u_c}{\partial W_{c,m}}$$

depends on the loss function

depends on the activation function

z_m
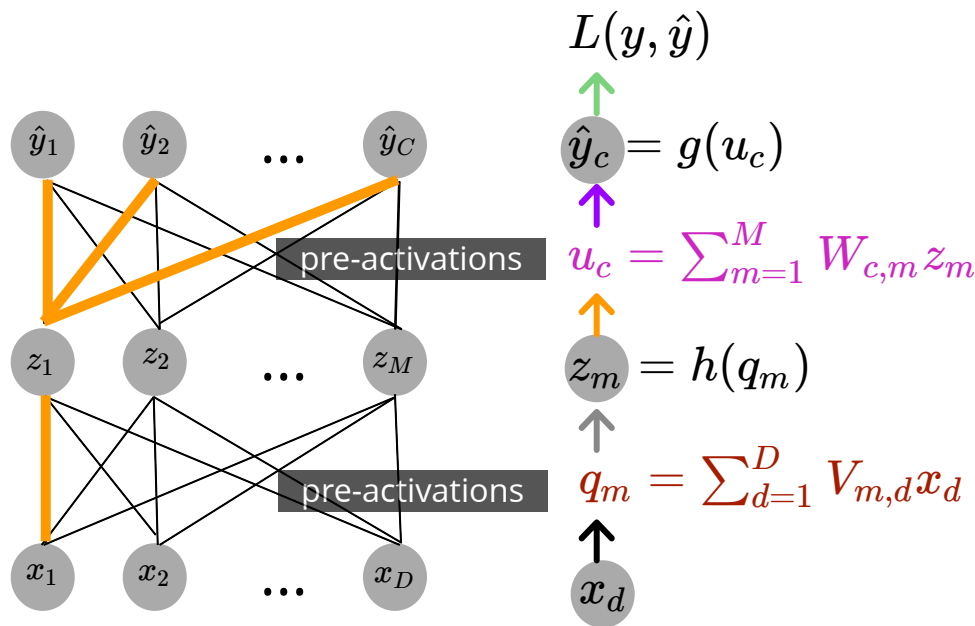
regression

C=1

$$L(y, \hat{y}) = \frac{1}{2}\|y - \hat{y}\|_2^2$$

$$\hat{y} = g(u) = u$$

combining the three terms above

$$\frac{\partial}{\partial W_m} L = (\hat{y} - y)z_m$$ we have seen this in linear regression lecture!

$L(y, \hat{y})$

$\hat{y}_c = g(u_c)$

$u_c = \sum_{m=1}^{M} W_{c,m} z_m$

$z_m = h(q_m)$

$q_m = \sum_{d=1}^{D} V_{m,d} x_d$

x_d

more generally:

$$\frac{\partial}{\partial W_{c,m}} L = (\hat{y}_c - y_c)z_m$$

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Gradient calculation

using the chain rule

$$\frac{\partial}{\partial W_{c,m}} L = \frac{\partial L}{\partial \hat{y}_c} \frac{\partial \hat{y}_c}{\partial u_c} \frac{\partial u_c}{\partial W_{c,m}}$$

☺

depends on the loss function

depends on the activation function

z_m

binary classification

scalar output C=1

$$\hat{y} = g(u) = \left(1 + e^{-u}\right)^{-1}$$

$$L(y, \hat{y}) = -y \log \hat{y} - (1 - y) \log(1 - \hat{y}) \qquad \frac{\partial \hat{y}}{\partial u} = \hat{y}(1 - \hat{y})$$

$$\frac{\partial}{\partial \hat{y}} L(y, \hat{y}) = -\frac{y}{\hat{y}} + \frac{(1 - y)}{(1 - \hat{y})}$$

combining the three terms above

$$\frac{\partial}{\partial W_m} L = (\hat{y} - y) z_m$$

looks familiar?
we had seen this in the logistic regression lecture

$L(y, \hat{y})$

$\hat{y}_c = g(u_c)$

$u_c = \sum_{m=1}^{M} W_{c,m} z_m$

$z_m = h(q_m)$

$q_m = \sum_{d=1}^{D} V_{m,d} x_d$

x_d

Gradient calculation

using the chain rule

$$\frac{\partial}{\partial W_{c,m}} L = \sum_{k=1}^{C} \frac{\partial L}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial u_c} \frac{\partial u_c}{\partial W_{c,m}}$$

☺

depends on the loss function

depends on the activation function

z_m

multiclass classification

C is the number of classes

$$L(y, \hat{y}) = -\sum_c y_c \log \hat{y}_c \quad \hat{y} = g(u) = \text{softmax}(u) \quad \text{softmax takes a vector and produces a vector}$$
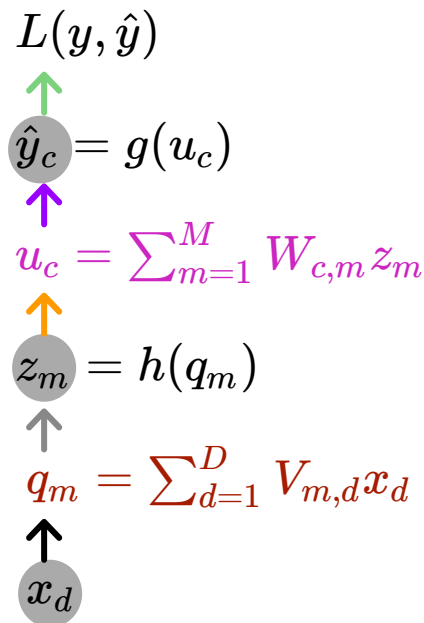
$$\frac{\partial}{\partial \hat{y}_k} L = -\frac{y_k}{\hat{y}_k}$$

$$\hat{y}_k = \frac{e^{u_k}}{\sum_i e^{u_i}} \quad \text{need to calculate the Jacobian} \quad \frac{\partial}{\partial u_c} \hat{y}_k = \begin{cases} \hat{y}_k (1 - \hat{y}_k) & k = c \\ -\hat{y}_c \hat{y}_k & k \neq c \end{cases}$$

combining the three terms above

$$\frac{\partial}{\partial W_{c,m}} L = (\hat{y}_c - y_c) z_m$$

again, this is familiar (softmax regression lecture)

$$L(y, \hat{y})$$

$$\uparrow$$

$$\hat{y}_c = g(u_c)$$

$$\uparrow$$

$$u_c = \sum_{m=1}^{M} W_{c,m} z_m$$

$$\uparrow$$

$$z_m = h(q_m)$$

$$\uparrow$$

$$q_m = \sum_{d=1}^{D} V_{m,d} x_d$$

$$\uparrow$$

$$x_d$$

Gradient calculation

gradient wrt V:

$$\frac{\partial}{\partial V_{m,d}} L = \sum_c \boxed{\frac{\partial L}{\partial \hat{y}_c} \frac{\partial \hat{y}_c}{\partial u_c}} \frac{\partial u_c}{\partial z_m} \frac{\partial z_m}{\partial q_m} \frac{\partial q_m}{\partial V_{m,d}}$$

☺

$$W_{c,m} \qquad\qquad x_d$$

depends on the middle layer activation

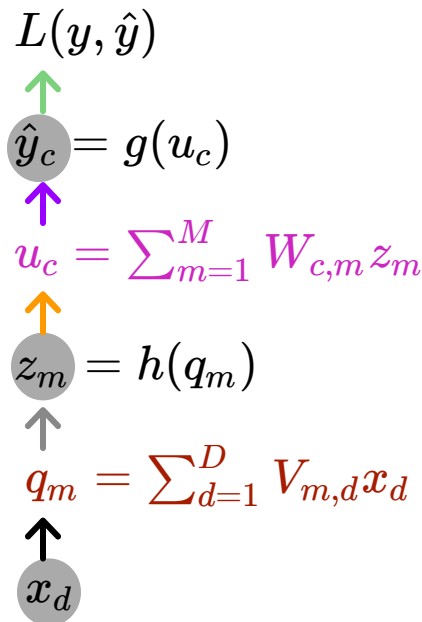|	
logistic function	$\sigma(q_m)(1 - \sigma(q_m))$
hyperbolic tan.	$1 - \tanh(q_m)^2$
ReLU	$\begin{cases} 0 & q_m \leq 0 \\ 1 & q_m > 0 \end{cases}$

example   logistic sigmoid

$$\frac{\partial}{\partial V_{m,d}} L = \sum_c (\hat{y}_c - y_c) W_{c,m} \sigma(q_m)(1 - \sigma(q_m)) x_d$$

$$= \sum_c (\hat{y}_c - y_c) W_{c,m} z_m (1 - z_m) x_d \qquad\qquad \Rightarrow \frac{\partial}{\partial V_{m,d}} J = \sum_n \sum_c (\hat{y}_c^{(n)} - y_c^{(n)}) W_{c,m} z_m^{(n)} (1 - z_m^{(n)}) x_d^{(n)}$$

$$L(y, \hat{y})$$

$$\hat{y}_c = g(u_c)$$

$$u_c = \sum_{m=1}^{M} W_{c,m} z_m$$

$$z_m = h(q_m)$$

$$q_m = \sum_{d=1}^{D} V_{m,d} x_d$$

$$x_d$$
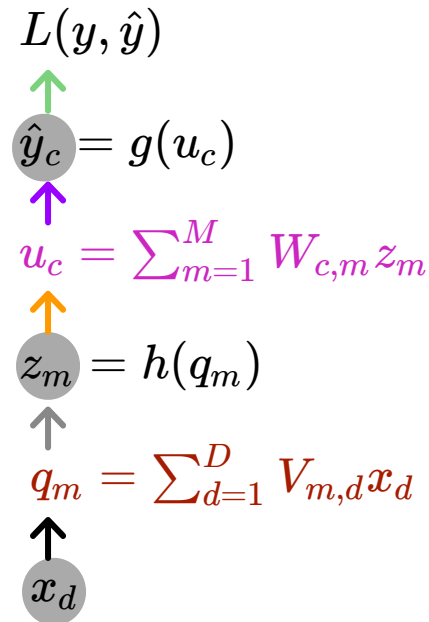
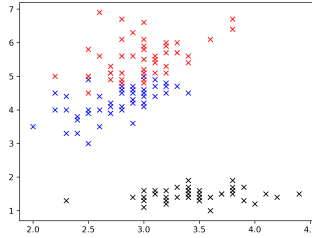for biases we simply assume the input is 1. $x_0^{(n)} = 1$

Gradient calculation

a common pattern

$$\frac{\partial}{\partial W_{c,m}} L = \boxed{\frac{\partial L}{\partial \hat{y}_c} \frac{\partial \hat{y}_c}{\partial u_c}} \boxed{\frac{\partial u_c}{\partial W_{c,m}}}$$

error from above $\frac{\partial L}{\partial u_c}$ input from below z_m

$$\frac{\partial}{\partial V_{m,d}} L = \boxed{\sum_c \frac{\partial L}{\partial \hat{y}_c} \frac{\partial \hat{y}_c}{\partial u_c} \frac{\partial u_c}{\partial z_m} \frac{\partial z_m}{\partial q_m}} \boxed{\frac{\partial q_m}{\partial V_{m,d}}}$$

error from above $\frac{\partial L}{\partial q_m}$ input from below x_d

$L(y, \hat{y})$

$\hat{y}_c = g(u_c)$

$u_c = \sum_{m=1}^{M} W_{c,m} z_m$

$z_m = h(q_m)$

$q_m = \sum_{d=1}^{D} V_{m,d} x_d$

x_d

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Example: classification

Iris dataset (D=2 features + 1 bias)

M = 16 hidden units

C=3 classes

cost is softmax-cross-entropy

```
1  def cost(x, #N x D
2           y, #N x C
3           w, #M x C
4           v, #D x M
5           ):
6    q = np.dot(x, v) #N x M
7    z = logistic(q) #N x M
8    u = np.dot(z, w) #N x C
9    yh = softmax(u)
10   nll = - np.mean(np.sum(u*y, 1) - logsumexp(u))
11   return nll
```
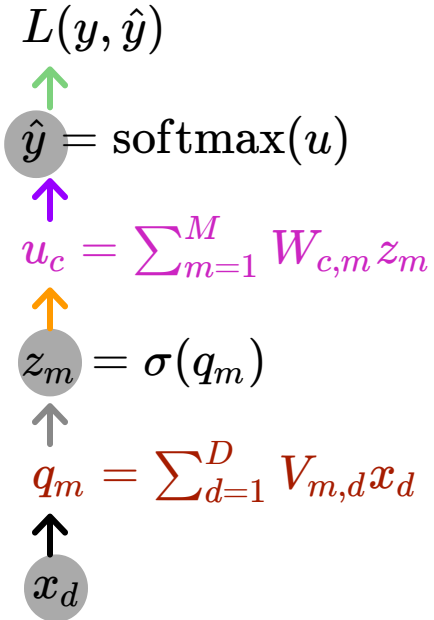
$L(y, \hat{y})$

$\hat{y} = \mathrm{softmax}(u)$

$u_c = \sum_{m=1}^{M} W_{c,m} z_m$

$z_m = \sigma(q_m)$

$q_m = \sum_{d=1}^{D} V_{m,d} x_d$

x_d

$$J = -\sum_{n=1}^{N} y^{(n)\top} u^{(n)} + \log \sum_c e^{u_c^{(n)}}$$

Example: classification

Iris dataset (D=2 features + 1 bias)

M = 16 hidden units

C=3 classes

$$L(y, \hat{y})$$

$$\hat{y} = \text{softmax}(u)$$

$$u_c = \sum_{m=1}^{M} W_{c,m} z_m$$

$$z_m = \sigma(q_m)$$

$$q_m = \sum_{d=1}^{D} V_{m,d} x_d$$
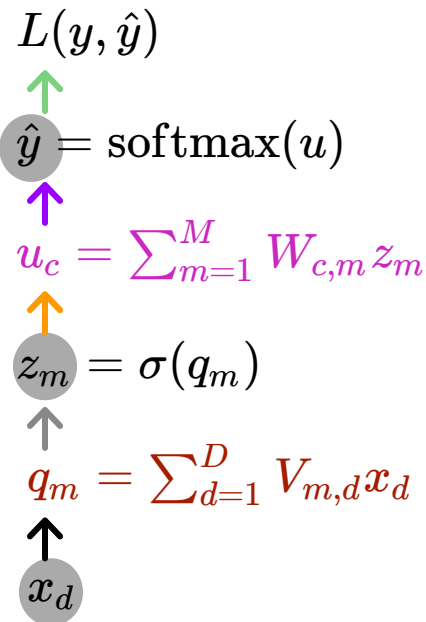
$$x_d$$
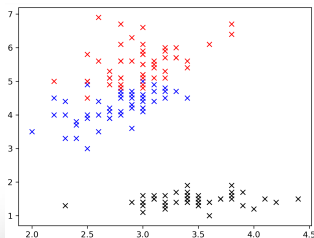
```
 1  def gradient(x,#N x D
 2               y,#N x C
 3               w,#M x C
 4               v,#D x M
 5               ):
 6      z = logistic(np.dot(x, v))#N x M
 7      N,D = x.shape
 8      yh = softmax(np.dot(z, w))#N x C
 9      dy = yh - y #N x C
10      dw= np.dot(z.T, dy)/N #M x C
11      dz = np.dot(dy, w.T) #N x M
12      dv = np.dot(x.T, dz * z * (1 - z))/N #D x M
13      return dw, dv
```

$$\frac{\partial}{\partial W_m} L = (\hat{y} - y) z_m$$

$$\frac{\partial}{\partial V_{m,d}} L = (\hat{y} - y) W_m z_m (1 - z_m) x_d$$

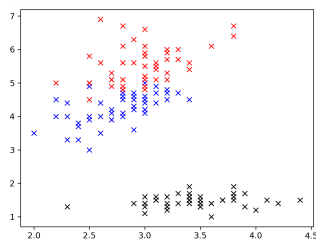check your gradient function using **finite difference** approximation that uses the *cost function*

```
 1  scipy.optimize.check_grad
```

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Example: classification

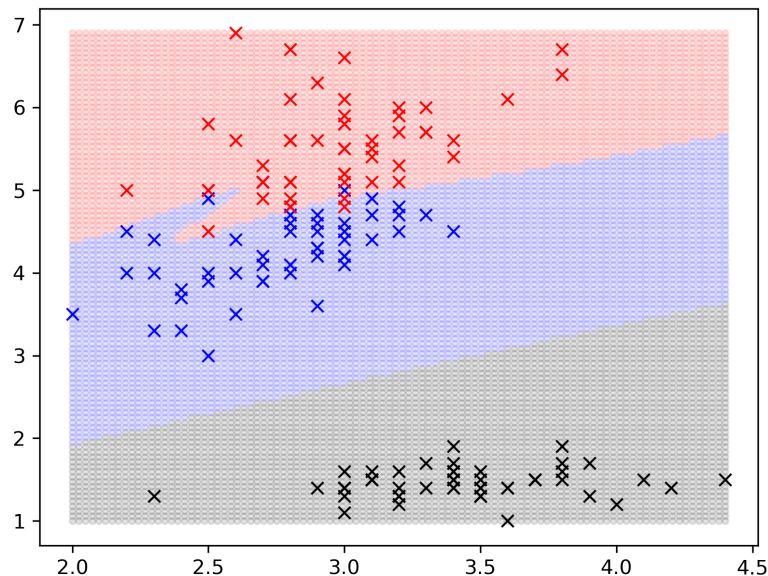

Iris dataset (D=2 features + 1 bias)

M = 16 hidden units

C=3 classes

```
1    while Condition:
2        dw, dv = gradient(x, y, w, v)
3        w = w - lr*dw
4        v = v - lr*dv
```

the resulting decision boundaries

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Automating gradient computation

gradient computation is tedious and mechanical. Can we automate it?

using **numerical differentiation**?

    approximates partial derivatives using finite difference $\frac{\partial f}{\partial w} \approx \frac{f(w+\epsilon) - f(w)}{\epsilon}$

    needs multiple forward passes (for each input output pair)

    can be slow and inaccurate

    useful for black-box cost functions or checking the correctness of gradient functions

symbolic differentiation: symbolic calculation of derivatives

    does not identify the computational procedure and reuse of values

automatic / algorithmic differentiation is what we want

    write code that calculates various functions, *e.g., the cost function*

    automatically produce (partial) derivatives *e.g., gradients used in learning*

Automatic differentiation

use the chain rule + derivative of simple operations $*, \sin, \frac{1}{x} \ldots$

use a computational graph as a data structure (for storing the result of computation)

step 1   break down to atomic operations

$$L = \tfrac{1}{2}(wx - y)^2 \longrightarrow$$

$$a_1 = w$$
$$a_2 = x$$
$$a_3 = y$$
$$a_4 = a_1 \times a_2$$
$$a_5 = a_4 - a_3$$
$$a_6 = a_5^2$$
$$a_7 = .5 \times a_6$$

step 2   build a graph with operations as internal nodes and input variables as leaf nodes

step 3   there are two ways to use the computational graph to calculate derivatives

forward mode: start from the leafs and propagate derivatives upward

reverse mode:

1. first in a bottom-up (forward) pass calculate the values $a_1, \ldots, a_4$
2. in a top-down (backward) pass calculate the derivatives

this second procedure is called **backpropagation** when applied to neural networks
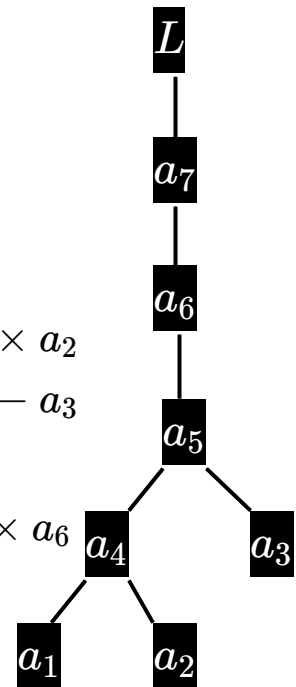
Forward mode

suppose we want the derivative $\frac{\partial y_1}{\partial w_1}$ where $\begin{cases} y_1 = \sin(w_1 x + w_0) \\ y_2 = \cos(w_1 x + w_0) \end{cases}$

we can calculate both y_1, y_2 and derivatives $\frac{\partial y_1}{\partial w_1}$ $\frac{\partial y_2}{\partial w_1}$ in a single forward pass

evaluation   **partial derivatives**

$$a_1 = w_0 \qquad \dot{a}_1 = 0$$
$$a_2 = w_1 \qquad \dot{a}_2 = 1$$
$$a_3 = x \qquad \dot{a}_3 = 0$$

we initialize these to identify which derivative we want

this means $\dot{\Box} = \frac{\partial \Box}{\partial w_1}$

$w_1 x$ $a_4 = a_2 \times a_3$ $\dot{a}_4 = a_2 \times \dot{a}_3 + \dot{a}_2 \times a_3$ x

$w_1 x + w_0$ $a_5 = a_4 + a_1$ $\dot{a}_5 = \dot{a}_4 + \dot{a}_1$ x

$y_1 = \sin(w_1 x + w_0)$ $a_6 = \sin(a_5)$ $\dot{a}_6 = \dot{a}_5 \cos(a_5)$ $x \cos(w_1 x + w_0) = \frac{\partial y_1}{\partial w_1}$

$y_2 = \cos(w_1 x + w_0)$ $a_7 = \cos(a_5)$ $\dot{a}_7 = -\dot{a}_5 \sin(a_5)$ $-x \sin(w_1 x + w_0) = \frac{\partial y_2}{\partial w_1}$

note that we get all partial derivatives $\frac{\partial \Box}{\partial w_1}$ in one forward pass

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Forward mode: computational graph

suppose we want the derivative $\dfrac{\partial y_1}{\partial w_1}$ where $\begin{cases} y_1 = \sin(w_1 x + w_0) \\ y_2 = \cos(w_1 x + w_0) \end{cases}$

we can represent this computation using a graph

once the nodes up stream calculate their values and derivatives we may discard a node

- *e.g.,* once $a_5, \dot{a}_5$ are obtained we can discard the values and partial derivatives for $a_4, \dot{a}_4, a_1, \dot{a}_1$

evaluation   **partial derivatives**

$$a_1 = w_0 \qquad\qquad \dot{a}_1 = 0$$

$$a_2 = w_1 \qquad\qquad \dot{a}_2 = 1$$

$$a_3 = x \qquad\qquad \dot{a}_3 = 0$$

$$a_4 = a_2 \times a_3 \qquad \dot{a}_4 = a_2 \times \dot{a}_3 + \dot{a}_2 \times a_3$$

$$a_5 = a_4 + a_1 \qquad \dot{a}_5 = \dot{a}_4 + \dot{a}_1$$

$$y_1 = a_6 = \sin(a_5) \qquad \dot{a}_6 = \dot{a}_5 \cos(a_5)$$

$$y_2 = a_7 = \cos(a_5) \qquad \dot{a}_7 = -\dot{a}_5 \sin(a_5)$$

$$y_1 = a_6 = \sin(a_5) \qquad\qquad\qquad y_2 = a_7 = \cos(a_5)$$
$$\tfrac{\partial y_1}{\partial w_1} = \dot{a}_6 = \dot{a}_5 \cos(a_5)\ \boxed{a_6} \qquad \boxed{a_7}\ \tfrac{\partial y_2}{\partial w_1} = \dot{a}_7 = -\dot{a}_5 \cos(a_5)$$

$$\boxed{a_5}\ \begin{array}{l} a_5 = a_4 + a_1 \\ \dot{a}_5 = \dot{a}_4 + \dot{a}_1 \end{array}$$

$$\begin{array}{r} a_4 = a_2 \times a_3 \\ \dot{a}_4 = a_2 \times \dot{a}_3 + \dot{a}_2 \times a_3 \end{array}\ \boxed{a_4} \qquad \boxed{a_1}\ \begin{array}{l} a_1 = w_0 \\ \dot{a}_1 = 0 \end{array}$$

$$\dot{a}_2 = 1\ \boxed{a_2} \qquad \boxed{a_3}\ \begin{array}{l} \dot{a}_3 = 0 \\ a_3 = x \end{array}$$

$$a_2 = w_1$$

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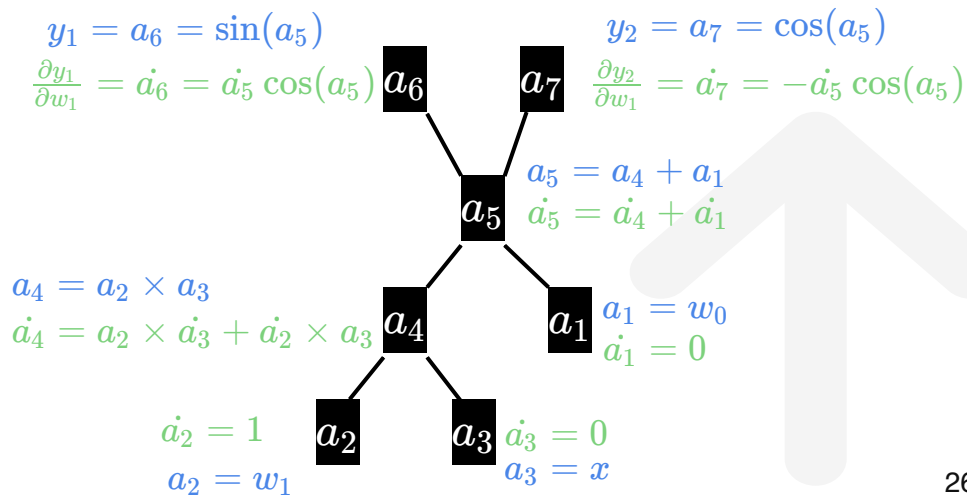
Reverse mode

suppose we want the derivative $\dfrac{\partial y_2}{\partial w_1}$ where $y_2 = \cos(w_1 x + w_0)$

first do a forward pass for evaluation

1) evaluation

$a_1 = w_0$

$a_2 = w_1$

then use these values to calculate partial derivatives in a backward pass

$a_3 = x$

$a_4 = a_2 \times a_3$

$a_5 = a_4 + a_1$

$w_1 x$

$w_1 x + w_0$

$y_1 = \sin(w_1 x + w_0)$ $\quad y_1 = a_6 = \sin(a_5)$

$y_2 = \cos(w_1 x + w_0)$ $\quad y_2 = a_7 = \cos(a_5)$

$\bar{a}_6 = 0 \;\; a_6 \qquad a_7 \;\; \bar{a}_7 = 1$

$a_5 \quad \bar{a}_5 = -\bar{a}_7 \sin(a_5) + \bar{a}_6 \cos(a_5)$

$\bar{a}_4 = \bar{a}_5 \;\; a_4 \qquad a_1 \;\; \bar{a}_1 = \bar{a}_5$

$\bar{a}_2 = a_3 \bar{a}_4 \;\; a_2 \qquad a_3 \;\; \bar{a}_3 = a_2 \bar{a}_4$

2) partial derivatives

$\dfrac{\partial y_2}{\partial a_7} = 1$ $\qquad \bar{a}_7 = 1$

$\dfrac{\partial y_2}{\partial a_6} = 0$ $\qquad \bar{a}_6 = 0$

$\left.\begin{array}{c}\\[2pt]\\[2pt]\end{array}\right\}$ this means $\bar{\Box} = \dfrac{\partial y_2}{\partial \Box}$
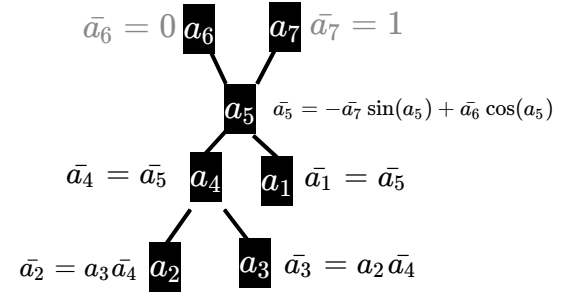
$\dfrac{\partial y_2}{\partial a_5} = \dfrac{\partial y_2}{\partial a_7}\dfrac{\partial a_7}{\partial a_5} + \dfrac{\partial y_2}{\partial a_6}\dfrac{\partial a_6}{\partial a_5} = -\sin(w_1 x + w_0)$ $\qquad \bar{a}_5 = -\bar{a}_7 \sin(a_5) + \bar{a}_6 \cos(a_5)$

$\dfrac{\partial y_2}{\partial a_4} = \dfrac{\partial y_2}{\partial a_5}\dfrac{\partial a_5}{\partial a_4} = -\sin(w_1 x + w_0)$ $\qquad \bar{a}_4 = \bar{a}_5$

$\dfrac{\partial y_2}{\partial x} = \dfrac{\partial y_2}{\partial a_3} = \dfrac{\partial y_2}{\partial a_4}\dfrac{\partial a_4}{\partial a_3} = -w_1 \sin(w_1 x + w_0)$ $\qquad \bar{a}_3 = \bar{a}_4 a_2$

$\dfrac{\partial y_2}{\partial w_1} = \dfrac{\partial y_2}{\partial a_2} = \dfrac{\partial y_2}{\partial a_4}\dfrac{\partial a_4}{\partial a_2} = -x \sin(w_1 x + w_0)$ $\qquad \bar{a}_2 = \bar{a}_4 a_3$

$\dfrac{\partial y_2}{\partial w_0} = \dfrac{\partial y_2}{\partial a_1} = \dfrac{\partial y_2}{\partial a_5}\dfrac{\partial a_5}{\partial a_1} = -\sin(w_1 x + w_0)$ $\qquad \bar{a}_1 = \bar{a}_5$

we get all partial derivatives $\dfrac{\partial y_2}{\partial \Box}$ in one backward pass

Reverse mode: computational graph

suppose we want the derivative $\dfrac{\partial y_2}{\partial w_1}$ where $y_2 = \cos(w_1 x + w_0)$

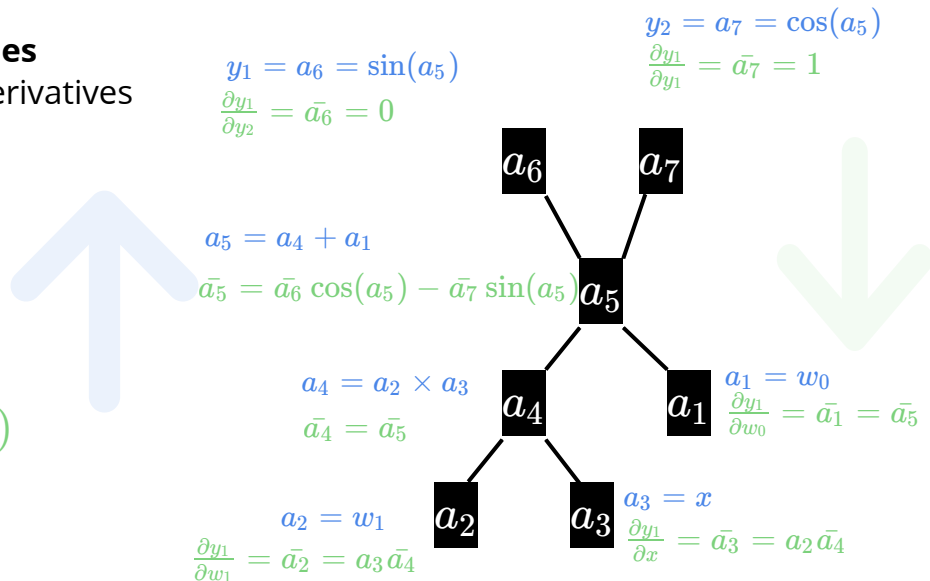we can represent this computation using a graph

1. in a forward pass we do evaluation and **keep the values**
2. use these values in the backward pass to get partial derivatives

1) evaluation

$a_1 = w_0$

$a_2 = w_1$

$a_3 = x$

$a_4 = a_2 \times a_3$

$a_5 = a_4 + a_1$

$y_1 = a_6 = \sin(a_5)$

$y_2 = a_7 = \cos(a_5)$

2) partial derivatives

$\bar{a}_7 = 1$

$\bar{a}_6 = 0$

$\bar{a}_5 = \bar{a}_6 \cos(a_5) - \bar{a}_7 \sin(a_5)$

$\bar{a}_4 = \bar{a}_5$

$\bar{a}_3 = a_2 \bar{a}_4$

$\bar{a}_2 = a_3 \bar{a}_4$

$\bar{a}_1 = \bar{a}_5$

$y_2 = a_7 = \cos(a_5)$
$\frac{\partial y_1}{\partial y_1} = \bar{a}_7 = 1$

$y_1 = a_6 = \sin(a_5)$
$\frac{\partial y_1}{\partial y_2} = \bar{a}_6 = 0$

$a_5 = a_4 + a_1$
$\bar{a}_5 = \bar{a}_6 \cos(a_5) - \bar{a}_7 \sin(a_5)$

$a_4 = a_2 \times a_3$
$\bar{a}_4 = \bar{a}_5$

$a_1 = w_0$
$\frac{\partial y_1}{\partial w_0} = \bar{a}_1 = \bar{a}_5$

$a_2 = w_1$
$\frac{\partial y_1}{\partial w_1} = \bar{a}_2 = a_3 \bar{a}_4$

$a_3 = x$
$\frac{\partial y_1}{\partial x} = \bar{a}_3 = a_2 \bar{a}_4$

$$\bar{a}_i = \frac{\partial y_c}{\partial a_i} = \sum_{j \in Parent(i)} \frac{\partial y_c}{\partial a_j} \frac{\partial a_j}{\partial a_i} = \sum_{j \in Parent(i)} \bar{a}_j \frac{\partial a_j}{\partial a_i}$$

Forward vs Reverse mode

forward mode is more natural, easier to implement and requires less memory

a single forward pass calculates $\frac{\partial y_1}{\partial w}, \ldots, \frac{\partial y_c}{\partial w}$

however, reverse mode is more efficient in calculating gradient $\nabla_w y = [\frac{\partial y}{\partial w_1}, \ldots, \frac{\partial y}{\partial w_D}]^\top$

this is more efficient if we have single output (cost) and many variables (weights)

for this reason, in training neural networks, reverse mode is used

the backward pass in the reverse mode is called **backpropagation**

many machine learning software implement autodiff:

- `autograd (extends numpy)`
- `pytorch`
- `tensorflow`

Improving optimization in deep learning

Initialization of parameters:

- random initialization (uniform or Gaussian) with small variance
  - break the symmetry of hidden units
- small positive values for bias (so that input to ReLU is >0)

models that are simpler to optimize:

this block is correcting for the residual error in the predictions of the previous layers

- using ReLU activation
- using **skip-connection** $x^{\{\ell+l\}} = \mathrm{ReLU}(W^{\{\ell+l\}}\mathrm{ReLU}(\ldots\mathrm{ReLU}(W^{\{\ell\}}x^{\{\ell\}})\ldots) + x^{\{\ell\}})$
- using **batch-normalization** (next)
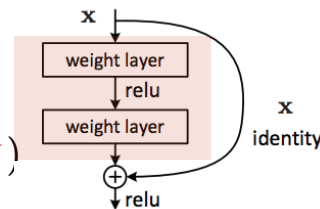
**Pretrain** a (simpler) model on a (simpler) task and

fine-tune on a more difficult target setting (has many forms)

continuation methods in optimization

- gradually increase the difficulty of the optimization problem
- good initialization for the next iteration

curriculum learning (similar idea)

- increase the number of "difficult" examples over time
- similar to the way humans learn

image credit: Mobahi'16

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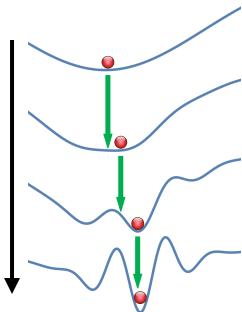
Batch Normalization

- gradient descent: parameters in all layers are updated
- distribution of inputs to layer ℓ changes
- each layer has to re-adjust
- inefficient for very deep networks

activation for the instance (n) at layer ℓ

$$\hat{x}_m^{\{\ell\},(n)} = \frac{x_m^{\{\ell\},(n)} - \mu_m^{\{\ell\}}}{\sigma_m^{\{\ell\}}}$$

unit m

idea   normalize the input to each unit (m) of a layer ℓ

alternatively: apply the batch-norm to $W^{\{\ell\}} x^{\{\ell\}}$

each unit is unnecessarily constrained to have zero-mean and std=1 (we only need to fix the distribution)

introduce learnable parameters $\mathrm{ReLU}(\gamma^{\{\ell\}} \mathrm{BN}(W^{\{\ell\}} x^{\{\ell\}}) + \beta^{\{\ell\}})$

- mean and std per unit is calculated for the minibatch during the forward pass
- we backpropagate through this normalization
- at test time use the mean and std. from the whole training set
- BN regularizes the model

recent observations   the change in distribution of activations is not a big issue empirically

BN works so well because it makes the loss function smooth

Summary

optimization landscape in neural networks is special and not yet fully understood

- exponentially many local optima and saddle points
- most local minima are good
- calculate the gradients using backpropagation

automatic differentiation

- simplifies gradient calculation for complex models
- gradient descent becomes simpler to use
- forward mode is useful for calculating the jacobian of $f : \mathbb{R}^Q \to \mathbb{R}^P$ when $P \geq Q$
- reverse mode can be more efficient when $Q > P$
  - backpropagation is reverse mode autodiff.

Better optimization in deep learning:

- better initialization
- models that are easier to optimize (using skip-connection, batch-norm, ReLU)
- pre-training and curriculum learning